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TRANSVERSE CORRUGATION SOLITONS IN A THREE-LAYERED NON-LINEARLY ELASTIC MEDIUM†

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The problem of the transverse bending deformations of a heavily loaded, thin, middle layer of material in a three-layer, linearly elastic medium is considered. Special versions of the theory of perturbations are developed to analyse the two-dimensional dynamics of the layers. As a result of "joining" them, a quasi-one-dimensional model is constructed which describes the evolution of the bendings of the middle layer close to the threshold of its stability. The possibility of the formation of "transverse corrugation" solitons which precede the inelastic deformation of the middle layer, is established. The condition for the existence of solitons are investigated as a function of the external stress, the thickness of the middle layer and the material parameters of the medium.

Undulating deformations of the individual layers of a material are observed experimentally for different methods of deforming sample [1]. Obviously, transverse corrugation accompanies the most heavily loaded layers of a medium with a restraining effect of the neighbouring layers, which are weakly loaded and therefore stable layers. In order to reveal the special features of this mechanism, we will consider the dynamics of a non-linearly elastic layer of material in the form of a plate, constricted by two half-spaces with smaller moduli of elasticity. The local bendings of the plate are assumed to be comparable with its thickness. The theory of finite deformation [2–4] is therefore used.

The Murnaghan theory of finite deformations is attractive due to the fact that, in this theory, the total non-linearly elastic energy of a system is chosen in the form of a series over all invariants of the Lagrangian strain tensor which are compatible with the symmetry of the medium. At the same time, more up-to-date versions of the non-linear theory of elasticity to simplify the problem are limited by a certain finite set of invariants, using additional geometrical hypotheses as a criterion for selecting them, the validity of which is difficult to estimate quantitatively. In the final analysis, each new "qualitative" choice of invariants must be tested on the solutions of actual dynamical problems.

A version of the reductive theory of perturbations is developed below. The advantage of this version is the fact that it enables one to pick out the main interactions, which reflect the dynamic symmetry of the problem under consideration, from the total non-linearly elastic energy of the system without prior hypotheses. The proposed procedure automatically leads to a reduction in the number of phenomenological constants in the initial expansion of the non-linearly elastic energy of the system, since these constants are confined by the conditions of self-consistency into a small number of parameters which will also be the experimentally observed effective moduli of elasticity of the layered medium.

The initial $(3 + 1)$-dimensional equations of the non-linear theory of elasticity are exceedingly complex to analyse. A constructive solution of the problem is possible by constructing simplified equations which correctly take account of the main features of the problem and, at the same time, can be solved exactly. The transverse bendings of a plate induce deformations in the underlying materials. The effects of the non-local counter effect of the substrata on the plate makes a theoretical description of its dynamics a lot more difficult. The problem is simplified close to the threshold of instability of the plate using linear theory. In this domain, it is possible to restrict the treatment to the slow space-time evolution of the linear mode which is responsible for the corrugation of the plate. Due to the instability of the

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linear mode, the non-linear properties of the medium manifest themselves and start to play a decisive role. The effects of non-linearity and dispersion limit the growth in the amplitude of the displacements and they open up the possibility of the formation of long-lived non-linear perturbations and structures in the plate.

The domain of characteristic space-time scales and external loads, in which an investigation of the corrugation of the plate using the simplified model is possible, is separated out. The boundary conditions for a three-layer medium, which correspondent to slippage of the middle layer past the supporting layers, are set up. The simplified model of the quasi-one-dimensional dynamics of the bendings of a heavily loaded layer of material is derived from the complete system of equations of the non-linear theory of elasticity, including all interactions which are compatible with the symmetry of the medium, with a controlled accuracy with respect to the small parameters, which reflect the magnitude of the external stress, the space-time response of the medium to external actions in the domain of characteristic scales which is being considered, as well as the geometrical and physical non-linearity of the material.

When constructing the model, a non-trivial, non-linear boundary-value problem is solved, in which the shape of the surface of the heavily loaded layer of the medium is not known in advance and is found in the process of solving the problem. We also mention that the proposed approach reduces the study of the real, non-unidimensional dynamics of a layered medium to an analysis of the solutions of the effective one-dimensional equations. The special features of the self-localized corrugation of the layer of the medium are determined as a result of the dispersion balance, which has a geometrical origin and depends on the thickness of the layer and the boundary conditions on its surface, the non-linear interaction of close unstable modes of deformation and, also, non-local interaction between the layers of the medium.

So far as we are aware, the dynamics of such deformations of a material have not been investigated, although self-localized, non-linearly elastic bendings of the individual layers of a medium leads to stress concentrators and, consequently, cause subsequent plastic flow of the material.

The possibility, in principle, of an analytical description of long-lived, spatially-localized, non-linearly elastic perturbations and of structures close to the thresholds of instability of multilayer media is illustrated. Solitons of the transverse corrugation of a layered medium, which precede the inelastic deformation of a material, are predicted and investigated on the basis of the model constructed.

Note that the model equations will be suitable for investigating the evolution of the shape of a heavily loaded layer and after it has lost stability, while the deformations remain non-linearly elastic and comparatively small.

1. FORMULATION OF THE PROBLEM. BASIC RELATIONS

Consider a plate of thickness d, constrained by two non-linearly elastic half-spaces, one of which is located above the plate ($x_3 \geq d/2$) and the other below the plate ($x_3 \leq -d/2$). Suppose $x_k(\check{x}_k)$ are the coordinates of a point mass of the plate (one of the supports) prior to deformation, $X_k = x_k + u_k(\mathbf{x}, t)$ ($\check{X}_k = \check{x}_k + \upsilon_k(\check{\mathbf{x}}, t)$) are the coordinates of the same point after deformation (the Latin subscripts take the values 1, 2, 3, unless otherwise stated), and $\mathbf{u}(\mathbf{x}, t)$ and $\mathbf{v}(\check{\mathbf{x}}, t)$ are displacement vectors.

In the theory of finite deformations [2–4], the elastic energy of a medium is described in the form of an expansion in the invariants of the Lagrange strain tensor

$$\eta_{ik} = \frac{1}{2}[\partial_i u_k + \partial_k u_i + \partial_i u_m \partial_k u_m], \quad \check{\eta}_{ik} = \frac{1}{2}[\partial_i \upsilon_k + \partial_k \upsilon_i + \partial_i \upsilon_m \partial_k \upsilon_m] \tag{1.1}$$

Suppose the material of the layers is isotropic. We take as independent invariants

$$I_1 = \eta_{ll}, \quad I_2 = \eta_{nm}^2, \quad I_3 = \eta_{nm}\eta_{mk}\eta_{kn}$$

We shall further denote the compatible physical quantities for the plate and the other layers using the same letters. We shall place a small concave arc (˘) over quantities which refer to the supports. When this does not give rise to any misunderstandings, we shall only talk about the plate.

We represent the energy of the non-linearly elastic plate in the form [2–4]

$$W = \int_{V_0} \phi d\mathbf{x}', \quad \phi = \sum_{n=2}^{\infty} \sum_{\langle kpq \rangle = n} A_{kpq} I_1^k I_2^p I_3^q \tag{1.2}$$

where ϕ is the energy referred to unit volume of the plate prior to deformation. The expression $\Sigma_{\langle kpq\rangle=n}$ denotes that the term for which $k+2p+3q=n(n\geq 2)$ are summed, and integration is carried out over the volume V_0 of the plate prior to deformation. We assume that A_{kpq} are comparable in order of magnitude.

The dynamic equations for the plate have the form [2–4]

$$-\rho_0\partial_t^2 u_i + \partial_s P_{is} = 0 \tag{1.3}$$

where

$$P_{is} = \frac{\partial\phi}{\partial[\partial_s u_i]} = \frac{\partial\phi}{\partial\eta_{is}} + \partial_k u_i \frac{\partial\phi}{\partial\eta_{ks}} \tag{1.4}$$

and ρ_0 = const is the density of the medium in the undeformed state.

The tensor P_{ij} is asymmetrical in the indices i, j. At the same time it is connected by a simple relation with the symmetrical stress tensor of the undeformed medium [2]

$$T_{ij} = \left[\det\left\|\frac{\partial\mathbf{X}}{\partial\mathbf{x}}\right\|\right]^{-1} P_{ik}\frac{\partial X_j}{\partial x_k} = \left[\det\left\|\frac{\partial\mathbf{X}}{\partial\mathbf{x}}\right\|\right]^{-1} P_{jk}\frac{\partial X_i}{\partial x_k} \tag{1.5}$$

For the subsequent analysis, it is convenient to introduce dimensionless variables. Suppose l is the characteristic scale of the deformations of the plate in the x_1Ox_2 plane, and a and $\tau_{ch} = l/\sqrt{\mu/\rho_0}$ are the characteristic amplitude and time of the deformations (μ is the shear modulus of the plate). We define two parameters $\epsilon_1 = a/l$ and $\epsilon_2 = d/l$ which reflect the order of smallness of the amplitudes of the displacements and the thickness of the plate. In the dynamic equations of the plate, we change to the dimensionless variables

$$\xi_\alpha = x_\alpha/l, \quad \eta = x_3/d, \quad \tau = t/\tau_{ch}, \quad u_i = a\bar{u}_i; \quad \alpha = 1, 2$$

They then take the form

$$\mu\epsilon_1\epsilon_2\partial_\tau^2\bar{u}_\alpha = \epsilon_2\partial_\beta P_{\alpha\beta} + \partial_\eta P_{\alpha 3}, \ \mu\epsilon_1\epsilon_2\partial_\tau^2\bar{u}_3 = \epsilon_2\partial_\beta P_{3\beta} + \partial_\eta P_{33}; \ \alpha, \beta = 1, 2; \ \partial_\alpha = \partial/\partial\xi_\alpha \tag{1.6}$$

We will now consider the domain of strong bending deformations of the plate, where the estimate $\epsilon_1 \sim \epsilon_2$ $(a \sim d)$ holds.

Suppose the support material has smaller moduli of elasticity compared with the material of the plate: $\breve{A}_{kpq}/A_{kpq} = O(\epsilon_1^3)$. An external stress T_{11}^{ext}, which is uniform at infinity, is applied only to the plate. Then

$$T_{11}^{\text{ext}}/\mu = O(\epsilon_1^2) + O(\epsilon_1^4)$$

The problem is simplified when there are no external loads of the order of ϵ_1^3[5].†

We will describe the deformations of the supports with other dimensionless variables:

$$\xi_k = \breve{x}_k/l, \quad \tau = t/\tau_{ch}, \quad \upsilon_i = a\bar{\upsilon}_i, \quad i, k = 1, 2, 3$$

In these dimensionless variables, the equations of the non-linear theory of elasticity for the supports have the form

$$\breve{\mu}\gamma_0\epsilon_1\partial_\tau^2\bar{\upsilon}_i = \partial_k\breve{P}_{ik}; \quad \gamma_0 = \mu\breve{\rho}_0/(\breve{\mu}\rho_0) = O(1), \quad \partial_i = \partial/\partial\xi_i \tag{1.7}$$

The domain of physical parameters of the problem, in which the non-linear dynamics of the plate can be studied within the framework of the simplified model, is separated out by the conditions which have been enumerated [5]. We next consider the case when the displacement fields of the three-layer medium are independent of the coordinate $x_2(\breve{x}_2)$ and the components of the displacements u_2 and υ_2

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are equal to zero. With the aim of constructing the model, we shall seek a solution of the dynamic equations of the three-layer medium in the form

$$\bar{u}_3 = \bar{u}_3^{(0)}(\xi_1,\tau) + \sum_{n=2}^{\infty} \bar{u}_3^{(n)}(\xi_1,\eta,\tau), \quad \bar{u}_1 = \bar{u}_1^{(0)}(\xi_1,\tau) + \sum_{n=1}^{\infty} \bar{u}_1^{(n)}(\xi_1,\eta,\tau)$$
$$\bar{\upsilon}_i = \sum_{n=0}^{\infty} \bar{\upsilon}_i^{(n)}(\xi_1,\xi_3,\tau), \quad i = 1,3 \tag{1.8}$$

The superscripts indicate the general order of the terms with respect to the parameters ϵ_1 and ϵ_2 ($\epsilon_1 \sim \epsilon_2$). Note that the fields $\bar{u}_3^{(0)}$ and $\bar{\upsilon}_i^{(0)}$ describe the local deformations of a medium with a characteristic scale l: $\partial\bar{\upsilon}_i^{(0)}/\partial\xi_k = O(1)$, $\partial\bar{u}_3^{(0)}/\partial\xi_1 = O(1)$ and, also, their slow space-time modulation, while the displacement $\bar{u}_1^{(0)}$ only describes the uniform planar stressed state of the plate and its slow modulations for which $\partial\bar{u}_1^{(0)}/\partial\xi_1 = O(\epsilon_1)$. These assertions will be refined below in Section 3 by introducing slow variables and, in particular, the separation of the term $\bar{u}_1^{(0)}$ from the correction $\bar{u}_1^{(1)}$ will be made specific.

The expansions of the tensors P_{ij} and $\breve{P}_{ij}$

$$P_{ij} = \sum_{n=1}^{\infty} P_{ij}^{(n)}, \quad \breve{P}_{ij} = \sum_{n=1}^{\infty} \breve{P}_{ij}^{(n)}; \quad i,j = 1,3 \tag{1.9}$$

correspond to the solution of the form (1.8), after they have been substituted into Eqs (1.6) and (1.7), a chain of perturbation-theory equations is obtained.

Analysis shows that, under the conditions formulate above, we can confine ourselves to a finite number of terms in representation (1.2) for the non-linearly elastic energy of the medium. We must retain the following terms in the energy density of the plate

$$\phi = \frac{\lambda}{2}I_1^2 + \mu I_2 + \frac{A}{3}I_3 + BI_1I_2 + \frac{C}{3}I_1^3 \tag{1.10}$$

where λ, μ, a and B, are the moduli of elasticity of the plate [3].

Since the supports have smaller moduli of elasticity, it is necessary to retain more terms in the energy of the supports to "join" the stresses along the interfaces of the media:

$$\phi = \frac{\breve{\lambda}}{2}\breve{I}_1^2 + \breve{\mu}\breve{I}_2 + \frac{\breve{A}}{3}\breve{I}_3 + \breve{B}\breve{I}_1\breve{I}_2 + \frac{\breve{C}}{3}\breve{I}_1^3 + \frac{\breve{H}}{4}\breve{I}_2^2 + \frac{\breve{F}}{6}\breve{I}_1\breve{I}_3 + \frac{\breve{M}}{2}\breve{I}_1^2\breve{I}_2 + \frac{\breve{N}}{12}\breve{I}_1^4 \tag{1.11}$$

The moduli of elasticity $\breve{H}, \breve{F}, \breve{M}, \breve{N}$, are introduced so that comparatively simple coefficients are obtained in the final formulae.

2. THE CONDITION FOR THE SLIPPAGE OF THE MIDDLE LAYER

We will now formulate the boundary conditions corresponding to slippage of the plate past the supports.

On the touching surfaces of the plate and the supports, the normal components of the displacements of the medium must be continuous. In the first orders of perturbation theory, the boundary conditions

$$\bar{u}_3^{(k)}\big|_{\eta=\pm 1/2} = \bar{\upsilon}_3^{(k)}\big|_{\xi_3^\pm = 0}, \quad k = 0,2; \quad \bar{\upsilon}_3^{(1)}\big|_{\xi_3^\pm = 0} = 0 \tag{2.1}$$

correspond to this requirement, where $\xi_3^\pm = \xi_3 \mp d/(2l)$.

The normal to the deformed surface of the plate is defined by the relation [2]

$$N_i = \frac{m_i}{|\mathbf{m}|}, \quad m_j = \frac{\partial x_s}{\partial X_j}n_s; \quad i,j = 1,3$$

where $\mathbf{n} = (0, 0, 1)$ is the vector of the normal to the undeformed surface of the plate. An expression for the normal in terms of the displacement fields, apart from terms of the order of ϵ_1^2, is next required, namely

$$N_1 = -\epsilon_1\partial_1\bar{u}_3^{(0)} + o(\epsilon_1^2), \quad N_3 = 1 - \frac{\epsilon_1^2}{2}(\partial_1\bar{u}_3^{(0)})^2 + o(\epsilon_1^2) \tag{2.2}$$

As a consequence of the slippage of the plate, the shear stresses in the deformed surfaces of the plate and the supports must vanish:

$$N_i e_{is} T_{sk} N_k = 0, \quad N_i e_{is} \breve{T}_{sk} N_k = 0$$

where e_{ij} is the antisymmetric unit tensor ($e_{13} = 1$). The expansion of these relations in the parameters ϵ_1, ϵ_2 (see formulae (1.5), (1.9) and (2.2)) gives the boundary conditions for the equations of perturbation theory

$$\begin{aligned} &P_{13}^{(4)}\big|_{\eta=\pm 1/2} = 0, \quad [P_{13}^{(k)} + \epsilon_1\partial_1\bar{u}_3^{(0)}P_{33}^{(k-1)}]\big|_{\eta=\pm 1/2} = 0, \quad k = 5, 6 \\ &\breve{P}_{13}^{(1)}\big|_{\xi_3^\pm=0} = 0, \quad [\breve{P}_{13}^{(n)} + \epsilon_1\partial_1\bar{u}_3^{(0)}\breve{P}_{33}^{(n-1)}]\big|_{\xi_3^\pm=0} = 0, \quad n = 2, 3 \end{aligned} \tag{2.3}$$

Note that, though the shear stresses at the touching surfaces of the plate and the supports vanish, the tangential components of the Piola–Kirchhoff tensors P_{13} and $\breve{P}_{13}$ do not vanish.

The requirement that the normal stresses should be continuous on the touching surfaces of the plate and the supports

$$N_i T_{ij} N_j = N_i \breve{T}_{ij} N_i$$

gives the following boundary conditions for the equations of perturbation theory

$$\begin{aligned} &P_{33}^{(4)}\big|_{\eta=\pm 1/2} = \breve{P}_{33}^{(1)}\big|_{\xi_3^\pm=0}, \quad P_{33}^{(5)}\big|_{\eta=\pm 1/2} = [\breve{P}_{33}^{(2)} - \epsilon_1\partial_1\bar{v}_1^{(0)}\breve{P}_{33}^{(1)}]\big|_{\xi_3^\pm=0} \\ &P_{33}^{(6)}\big|_{\eta=\pm 1/2} = \{\breve{P}_{33}^{(3)} - \epsilon_1\partial_1\bar{v}_1^{(0)}\breve{P}_{33}^{(2)} + \epsilon_1\breve{P}_{33}^{(1)}[\partial_1\bar{u}_1^{(1)} - \partial_1\bar{v}_1^{(1)}]\}\big|_{\xi_3^\pm=0} \end{aligned} \tag{2.4}$$

In considering relations containing the components of $P_{ij}^{(n)}$ and $\breve{P}_{ij}^{(k)}$, it is necessary to take account of the different order of magnitude of the moduli of elasticity of the plate and the supports.

The boundary conditions on the edges of the plate characterize the counter effect on the plate of the deformations in the supports. In particular, from conditions (2.4), this effect takes account of the change in the normal stresses as a consequence of the different compressibility of the plate and support materials.

In formulae (2.3) and (2.4), due to the small moduli of elasticity of the supports, the link between the stresses on the surface of the plate and the stresses in the supports manifested itself starting from the fourth order of perturbation theory for the plate. The stresses on the developed surface of the plate vanish in the lower orders of perturbation theory. This requirement is equivalent to the conditions

$$P_{i3}^{(n)}\big|_{\eta=\pm 1/2} = 0, \quad n = 1, 2, 3 \tag{2.5}$$

The static deformations of the medium at infinity must ensure the equilibrium of the supports which have been compressed on account of the deformation of the plate. The dynamic deformations of the supports vanish at infinity.

3. THE REDUCTIVE THEORY OF PERTURBATIONS

The non-linear dynamics of the plate are determined not only by the local interactions of its own deformations but, also, by the "indirect" interaction of the deformations of the plate in terms of the displacements of the supports. Even in the simplest case of a quasi-one-dimensional corrugation of a plate, the displacements of the supports are two-dimensional. The indirect interactions are therefore non-local. In the case of the quasi-one-dimensional dynamics of a plate, perturbation theory (1.8) gives, generally speaking, non-linear, integrodifferential equations. The problem is simplified and reduces to an "almost local" problem when the plate experiences undulating corrugation close to that which corresponds to a neutral-stable linear mode.

We shall consider this case. According to linear theory, close to the threshold of instability of the plate, it is necessary to specify the solution (1.8) by introducing the dependence on the slow variables X and T

$$\bar{u}_1 = \bar{u}_1^{(0,0)}(X,T) + \sum_{n=1}^{\infty}\sum_{l=-\infty}^{\infty} \bar{u}_1^{(n,l)}(X,T,\eta)\exp(ikl\xi_1)$$

$$\bar{u}_3 = [\bar{u}_3^{(0,1)}(X,T)\exp(ik\xi_1) + \text{c.c.}] + \sum_{n=2}^{\infty}\sum_{l=-\infty}^{\infty} \bar{u}_3^{(n,l)}(X,T,\eta)\exp(ikl\xi_1) \tag{3.1}$$

$$\bar{v}_m = \sum_{n=1}^{\infty}\sum_{l=-\infty}^{\infty} \bar{v}_m^{(n,l)}(X,T,\xi_3)\exp(ikl\xi_1), \quad m = 1, 3$$

The index n characterizes the order of the terms with respect to the parameters ϵ_1, ϵ_2 ($\epsilon_1 \sim \epsilon_2$), and k is the wave number of the neutral-stable linear mode which is formed in the case of the critical stress T_{11}^{lin}. The values of k and T_{11}^{lin} are found when solving the problem. Further analysis shows that the estimate $T_{11}^{\text{lin}}/\mu = O(\epsilon_1^2)$ holds.

In the case of external stresses T_{11}^{ext} close to T_{11}^{lin}, the non-linear dynamics of the plate is determined by the unstable modes, the wave numbers of which lie in a small neighbourhood of the critical wave number k. The radius of the neighbourhood depends on the extent to which the stress T_{11}^{ext} differs from T_{11}^{lin}. We will henceforth assume that $(T_{11} - T_{11}^{\text{lin}})/\mu = O(\epsilon_1^4)$, and then the slow variables: $X = \epsilon_1\xi_1$, $T = \epsilon_1^2\tau$, which describe the modulations of the fundamental harmonic $\sim\exp(ik\xi_1)$ as a result of its interaction with the close unstable modes, are defined in terms of the parameter ϵ_1. In the final analysis, the validity of the scale expansion is justified by the self-consistency of the results.

The versions of perturbation theory are considered in accordance with relations (3.1). One of them is for the description of the non-linear dynamics of the plate and the other is for the description of the non-linear dynamics of the supports. As a result of their "joining", an effective, quasi-one-dimensional model of the evolution of the envelop of the transverse bendings of the plate will be constructed.

Perturbation theory for the plate. The chosen form of the solution (3.1) leads to the following representations for the components of the tensors η_{sp} and P_{sp}

$$\eta_{sp} = \sum_{n=1}^{\infty}\sum_{l=-\infty}^{\infty} \eta_{sp}^{(n,l)}\exp(ikl\xi_1), \quad P_{sp} = \sum_{n=1}^{\infty}\sum_{l=-\infty}^{\infty} P_{sp}^{(n,l)}\exp(ikl\xi_1) \tag{3.2}$$

We find the relation between the coefficients $\eta_{ij}^{(n,l)}$ and $\bar{u}_i^{(n,l)}$ by substituting expression (3.1) and (3.2) into relations (1.1). Since the initial fields are real, the functions $\bar{u}_i^{(n,l)}$, etc. satisfy the conditions

$$\bar{u}_i^{(n,-l)} = [\bar{u}_i^{(n,l)}]^*$$

The components $\bar{u}_i^{(n,l)}$, $\eta_{ij}^{(n,l)}$, $P_{ij}^{(n,l)}$ depend on the variables η, X and T. Substituting expressions (3.2) into relations (1.6), (2.1), (2.3)–(2.5) and equating terms of the first order of smallness for each of the independent harmonics, we obtain boundary-value problems in the variable η.

In the first orders of perturbation theory, we have boundary-value problems with trivial solutions: $P_{13}^{(n,l)} \equiv 0$ ($n = 1, 2$) and $P_{33}^{(k,l)} \equiv 0$ ($k = 2, 3$). The first of these is equivalent to the condition $\eta_{13}^{(n,l)} = 0$ from which it firstly follows that the coefficients $\eta_{13}^{(n,l)}$, in fact, are not equal to zero starting from $n = 3$. If the condition $\eta_{13}^{(n,l)} = 0$ ($n = 1, 2$) is rewritten in terms of the displacements, equations for calculating the fields $\bar{u}_1^{(n,l)}$ are obtained. The functions $\bar{u}_1^{(n,l)}$ ($n = 1, 2$) with $l = 0, 2, 3, \ldots$ turn out to be independent of η. We shall subsequently indicate functions which are independent of η with a tilde: $\bar{u}_1^{(n,l)} = \tilde{u}_1^{(n,l)}$ ($n = 1, 2$; $l = 0, 2, 3, \ldots$). The coefficients $\bar{u}_1^{(n,1)}$ ($n = 1, 2$) are expressed in terms of $\tilde{u}_3^{(0,1)}$

$$\bar{u}_1^{(1,1)} = -\epsilon_2 ik\eta\tilde{u}_3^{(0,1)} + \tilde{u}_1^{(1,1)}, \quad \bar{u}_1^{(2,1)} = -\epsilon_1\epsilon_2\eta\partial_X\tilde{u}_3^{(0,1)} + \tilde{u}_1^{(2,1)} \tag{3.3}$$

where $\tilde{u}_1^{(n,1)}$ ($n = 1, 2$) are arbitrary functions which arise during integration; they are determined by the following orders of perturbation theory. It is found that $\tilde{u}_1^{(n,1)} = 0$ and ($n = 1, 2$).

From the relations

$$P_{33}^{(n,l)} = (\lambda+2\mu)\eta_{33}^{(n,l)} + \lambda\eta_{11}^{(n,l)} \equiv 0, \quad n = 2, 3 \tag{3.4}$$

we find the connection between $\eta_{11}^{(n,l)}$ and $\eta_{33}^{(n,l)}$ which is useful in the subsequent calculations and leads to the representation

$$P_{11}^{(n,l)} = (\lambda'+2\mu)\eta_{11}^{(n,l)}, \quad n = 2, 3$$

Henceforth, the effective modulus of elasticity of planar deformation

$$\lambda' = 2\lambda\mu/(\lambda+2\mu)$$

is introduced which characterizes the stresses (in the linear approximation, see [5]) which arise in the middle layer when an element of its area is changed.

We will now explain the general scheme for the calculations by taking as an example the boundary-value problem

$$\partial_\eta P_{13}^{(3,l)} + \epsilon_2 iklP_{11}^{(2,l)} = 0; \quad P_{13}^{(3,l)}\Big|_{\eta=\pm 1/2} = 0, \quad l = 0, 1, 2, \dots \tag{3.5}$$

We integrate Eq. (3.5) over the thickness of the plate. We obtain the condition for it to be solvable

$$\int_{-1/2}^{1/2} P_{11}^{(2,l)}d\eta = (\lambda'+2\mu)\int_{-1/2}^{1/2}\eta_{11}^{(2,l)}d\eta = 0, \quad l = 1, 2, \dots$$

from which we find the relation between the functions

$$\tilde{u}_1^{(1,1)} = 0, \quad \bar{u}_1^{(1,2)} = \tilde{u}_1^{(1,2)} = -\frac{\epsilon_1 ik}{4}(\tilde{u}_3^{(0,1)})^2, \quad \bar{u}_1^{(1,l)} = 0, \quad l = 3, 4, \dots \tag{3.6}$$

which have already been introduced.

The constraints (3.6) mean that only two of the coefficients $\eta_{11}^{(2,l)}$ $(l \ge 0)$

$$\eta_{11}^{(2,0)} = \epsilon_1^2(\partial_X\tilde{u}_1^{(0,0)} + k^2|\tilde{u}_3^{(0,1)}|^2) \equiv \epsilon_{11}^{(2,0)}, \quad \eta_{11}^{(2,1)} = \epsilon_1\epsilon_2 k^2\eta\tilde{u}_3^{(0,1)} \tag{3.7}$$

do not vanish.

The quantity $\epsilon_{11}^{(2,0)}$ characterizes the longitudinal deformation of the plate, which is homogeneous throughout its thickness.

When $l = 0, 1, 2, \dots$, the solutions of problem (3.5) have the form

$$P_{13}^{(3,1)} = -\frac{1}{2}(\lambda'+2\mu)\epsilon_1\epsilon_2^2 ik^3\tilde{u}_3^{(0,1)}\left(\eta^2-\frac{1}{4}\right), \quad P_{13}^{(3,l)} = 0, \quad l = 0, 2, 3, \dots \tag{3.8}$$

The constraints on the quantities $\eta_{ij}^{(2,l)}$ which have been established (see (3.7)) enable us to turn to Eqs (3.4) and solve them for the fields $\bar{u}_3^{(2,l)}$. As a result of integrating the equations obtained, the corrections $\bar{u}_3^{(2,l)}$ are calculated

$$\begin{aligned}
\frac{a}{d}\bar{u}_3^{(2,0)} &= -\frac{\lambda'}{2\mu}\epsilon_{11}^{(2,0)}\eta - (\epsilon_1 k)^2|\tilde{u}_3^{(0,1)}|^2\eta + \frac{a}{d}\tilde{u}_3^{(2,0)} \\
\bar{u}_3^{(2,1)} &= -\frac{\lambda'}{4\mu}(\epsilon_2 k\eta)^2\tilde{u}_3^{(0,1)} + \tilde{u}_3^{(2,1)} \\
\bar{u}_3^{(2,2)} &= \frac{\epsilon_1\epsilon_2 k^2}{2}(\tilde{u}_3^{(0,1)})^2\eta + \tilde{u}_3^{(2,2)} \\
\bar{u}_3^{(2,l)} &= \tilde{u}_3^{(2,l)}, \quad l = 3, 4, \dots
\end{aligned} \tag{3.9}$$

where $\tilde{u}_3^{(2,l)}$ $(l = 0, 1, \dots)$ are functions which arise in the integration.

In order to progress further, we note that, on the one hand, the components $P_{13}^{(3,l)}$ have already been found in (3.8) and, on the other, they can be expressed in terms of the deformations

$$P_{13}^{(3,l)} = 2\mu\eta_{13}^{(n,l)} \tag{3.10}$$

Not only the deformations $\eta_{13}^{(3,l)}$, but also the longitudinal displacements $\bar{u}_1^{(3,l)}$

$$\bar{u}_1^{(3,0)} = \tilde{u}_1^{(3,0)}$$

$$\bar{u}_1^{(3,1)} = -\epsilon_2^3 ik^3\left[\left(1+\frac{\lambda'}{4\mu}\right)\frac{\eta^3}{3} - \frac{1}{4}\left(1+\frac{\lambda'}{2\mu}\eta\right)\right]\tilde{u}_3^{(0,1)} + \epsilon_2 ik\eta\left[-\tilde{u}_3^{(2,1)} + \left(1+\frac{\lambda'}{4\mu}\right)\epsilon_{11}^{(2,0)}\tilde{u}_3^{(0,1)}\right] + \tilde{u}_1^{(3,1)}$$

$$\bar{u}_1^{(3,2)} = \frac{\lambda'}{4\mu}\epsilon_1\epsilon_2^2 ik^3\eta^2[\tilde{u}_3^{(0,1)}]^2 - 2\epsilon_2 ik\eta\tilde{u}_3^{(2,2)} + \tilde{u}_1^{(3,2)} \tag{3.11}$$

$$\partial_\eta\bar{u}_1^{(3,l)} + \epsilon_2 ikl\tilde{u}_3^{(2,l)} = 0, \quad l = 3, 4\ldots$$

can be found from Eqs (3.10).

The arbitrary functions $\tilde{u}_1^{(3,l)}$ appear again after integrating relations (3.10).

The overall computational scheme will be self-consistent, if the functions, which were arbitrary in the first orders of perturbation theory, are, in the final analysis, united into a block such that a closed system of equations is obtained which determines the evolution of the first of them. This system of equations will also be an effective model of the non-linear dynamics of the plate. The procedure is closed only if the slow variables are correctly chosen. The relations between the functions arise from the conditions for the boundary-value problems of perturbation theory to be solvable. It is noteworthy that the proposed perturbation theory satisfies the criterion which has been formulated. The necessary calculations are simple but tedious and are carried out using the scheme which has already been described. We will enumerate the key aspects.

For the problem

$$\partial_\eta P_{13}^{(4,l)} + \epsilon_2 iklP_{11}^{(3,l)} + \epsilon_1\epsilon_2\partial_X P_{11}^{(2,l)} = 0; \quad P_{13}^{(4,l)}\big|_{\eta=\pm1/2} = 0, \quad l = 0, 1, 2, \ldots \tag{3.12}$$

the conditions of solvability not only give algebraic relations between the functions which have arisen during the integrations but, also, the equation

$$\partial_X\sigma_{11}^{(2,0)} = 0, \quad \sigma_{11}^{(2,0)} = (\lambda' + 2\mu)\epsilon_{11}^{(2,0)} \tag{3.13}$$

Since the external stress at infinity $[T_{11}^{\text{ext}}]^{(2,0)} = \text{const}$, we conclude from relations (1.5) and (3.13) that

$$\sigma_{11}^{(2,0)} = (\lambda' + 2\mu)\epsilon_1^2[\partial_X\tilde{u}_1^{(0,0)} + k^2|\tilde{u}_3^{(0,1)}|^2] = [T_{11}^{\text{ext}}]^{(2,0)} = \text{const} \tag{3.14}$$

Using the constraints which have been found, we calculate the components $P_{13}^{(4,l)}$ from the solution of problem (3.12). Only $P_{13}^{(4,1)} = -i\epsilon_1\partial_X\partial_k P_{13}^{(3,1)}$ is found not to be equal to zero. This information about $P_{13}^{(4,l)}$ is sufficient to construct an effective model. It is not possible to calculate the corrections $\bar{u}_1^{(4,l)}$ since they do occur in the equations for the bending of the plate. The fields $\bar{u}_3^{(3,l)}$ also do not appear in the effective equations and it is therefore possible to avoid the integration of system (3.4) when $n = 3$.

The reactions of the supports to the bending of the plate manifest themselves starting from the fourth order of perturbation theory.

We will illustrate this, taking the following boundary-value problem as an example

$$\epsilon_2 iklP_{31}^{(3,l)} + \partial_\eta P_{33}^{(4,l)} = 0; \quad P_{33}^{(4,l)}\big|_{\eta=\pm1/2} = \check{P}_{33}^{(1,l)}\big|_{\xi_3^\pm=0} \tag{3.15}$$

Stress $\check{P}_{33}^{(1,l)}|_{\xi_3^\pm=0}$, which are due to deformations of the supports, act on the plate surfaces $\eta = \pm1/2$. The version of perturbation theory (for the plate and for the supports) are therefore interrelated. The success of the method is due to the fact that the boundary-value problems for the supports are solved using displacements and deformations which have already been calculated on the plate surfaces $\eta = \pm 1/2$. In order not to interrupt the analysis of the dynamics of the plate, we will consider the perturbation theory for the supports in the following section, and it is here that we present the results of the solution of the boundary-value problems for the supports. Only the component $\breve{P}_{33}^{(1,l)}$ is non-zero on the plate boundary and it is defined in terms of the transverse displacements of the plate $\tilde{u}_3^{(0,1)}$.

$$\breve{P}_{33}^{(1,1)}\Big|_{\xi_3^{\pm}=0} = \mp\frac{1}{2}(\breve{\lambda}' + 2\breve{\mu})\epsilon_1|k|\tilde{u}_3^{(0,1)}(X,T), \quad \breve{\lambda}' = \frac{2\breve{\mu}\breve{\lambda}}{\breve{\lambda} + 2\breve{\mu}} \tag{3.16}$$

Note also that, when $n \geq 3$, the fields $P_{13}^{(n,l)}$ and $P_{31}^{(n,l)}$ are not equal. However, according to relations (1.4), they are related to one another so that it is always possible to find $P_{31}^{(n,l)}$ using the known components $P_{13}^{(n,l)}$. In particular, using relations (3.8), it can be shown that, when $l \geq 0$, two of the functions $P_{31}^{(3,l)}$

$$\begin{aligned} P_{31}^{(3,1)} &= -\frac{1}{2}(\lambda' + 2\mu)\epsilon_1\epsilon_2^2 ik^3\tilde{u}_3^{(0,1)}\left(\eta^2 - \frac{1}{4}\right) + \epsilon_1 ik\tilde{u}_3^{(0,1)}[T_{11}^{\text{ext}}]^{(2,0)} \\ P_{31}^{(3,2)} &= (\lambda' + 2\mu)\epsilon_1^2\epsilon_2 ik^3[\tilde{u}_3^{(0,1)}]^2\eta \end{aligned} \tag{3.17}$$

are non-zero.

On integrating Eq. (3.15) with $l = 1$ over the thickness of the plate, we obtain the condition for it to be solvable

$$[T_{11}^{\text{ext}}]^{(2,0)} = -\frac{\breve{\lambda}' + 2\breve{\mu}}{\epsilon_2|k|} - \frac{(\epsilon_2 k)^2}{12}(\lambda' + 2\mu) \tag{3.18}$$

Relation (3.18) connects the external compressive stress $[T_{11}^{\text{ext}}]^{(2,0)}$ with the wave vector k of the neutral stable linear mode, which is responsible for the corrugation of the plate. From the condition for an extremum of the function $[T_{11}^{\text{ext}}]^{(2,0)}(k)$, we find the minimum stress T_{11}^{lin} and strain $\epsilon_{11}^{(2,0)}$, starting from which corrugation of the plate is observed, and the value of the wave number k_0 corresponding to it.

$$T_{11}^{\text{lin}} = -\frac{\lambda' + 2\mu}{4}(\epsilon_2 k_0)^2 = (\lambda' + 2\mu)\epsilon_{11}^{(2,0)}, \quad \epsilon_2^3|k_0|^2 = 6\frac{\breve{\lambda}' + 2\breve{\mu}}{\lambda' + 2\mu} \tag{3.19}$$

The solutions of the fifth-order equations of perturbation theory give the fields $P_{13}^{(5,l)}$ and $P_{33}^{(5,l)}$. The component $P_{31}^{(5,1)}$ is expressed in terms of $P_{13}^{(5,1)}$ and the other known fields using relation (1.4). In order to construct a model of the bending of the plate, $P_{31}^{(5,1)}$ is required, though not the function itself but its mean value. We now present the final result

$$\begin{aligned} \int_{-1/2}^{1/2} P_{31}^{(5,1)}(\eta)d\eta &= \epsilon_1 ik_0\tilde{u}_3^{(0,1)}[\sigma_{11}^{(4,0)} + (\epsilon_2 k_0)^4 p] - \\ &- \frac{\lambda' + 2\mu}{6}\epsilon_1\epsilon_2^2 ik_0^3\tilde{u}_3^{(2,1)} - \frac{\lambda' + 2\mu}{4}\epsilon_1^3\epsilon_2^2 ik_0\partial_X^2\tilde{u}_3^{(0,1)} + \epsilon_1\frac{g_v^{(4)}}{ik_0}|\tilde{u}_3^{(0,1)}|^2\tilde{u}_3^{(0,1)} \end{aligned} \tag{3.20}$$

where

$$p = \frac{1}{8}\left[\left(1 + \frac{\lambda'}{2\mu}\right)\left(\frac{\lambda'}{6} - \frac{3}{5}\mu\right) + \frac{3a_1 + a_2}{6}\right]$$

a_1 and a_2 are the effective moduli of elasticity of the plate, introduced earlier [5], where

$$3a_1 + a_2 = (A + 2B)\left[1 - \left(\frac{\lambda'}{2\mu}\right)^3\right] + (B + C)\left[1 - \frac{\lambda'}{2\mu}\right]^3$$

The parameter

$$g_v^{(4)} = \frac{(\epsilon_1\epsilon_2 k_0^3)^2}{2}\left[\frac{1}{2}\left(1 + \frac{\lambda'}{2\mu}\right)\left(\lambda' - \frac{5}{3}\mu\right) - \frac{3a_1 + a_2}{3}\right]$$

characterizes the interaction of the transverse modes in the bulk of the plate. The quantity $\sigma_{11}^{(4,0)} = (\lambda' + 2\mu)\epsilon_{11}^{(4,0)}$ is defined in terms of the part $\epsilon_{11}^{(4,0)}$ of the longitudinal strain tensor $\eta_{11}^{(4,0)}$, which is homogeneous throughout the thickness of the plate and is a combination of functions which were arbitrary in the first orders of perturbation theory. In order to construct an effective model, there is no need to represent $\sigma_{11}^{(4,0)}$ in terms of the displacement fields, since the conditions for the sixth-order equations of perturbation theory to be solvable give a closed system for calculating $\sigma_{11}^{(4,0)}$, $\tilde{u}_3^{(1,0)}$ and $\tilde{u}_1^{(0,0)}$.

Two of the sixth-order equations of perturbation theory are used to construct the model of the bending of the plate

$$\begin{aligned}\mu\epsilon_2\epsilon_l^5\partial_T^2\tilde{u}_l^{(0,0)} &= \partial_\eta P_{13}^{(6,0)} + \epsilon_l\epsilon_2\partial_X P_{1l}^{(4,0)}\\ \mu\epsilon_2\epsilon_l^5\partial_T^2\tilde{u}_3^{(0,1)} &= \partial_\eta P_{33}^{(6,1)} + \epsilon_2 ik_0 P_{31}^{(5,1)} + \epsilon_l\epsilon_2\partial_X P_{31}^{(4,1)}\end{aligned} \tag{3.21}$$

The functions $P_{13}^{(6,0)}$ and $P_{33}^{(6,1)}$ are unknown in Eqs (3.21).

The boundary condition

$$P_{13}^{(6,0)}\Big|_{\eta=\pm 1/2} = 0 \tag{3.22}$$

follows from conditions (2.3).

The procedure for calculating the boundary values $P_{33}^{(6,1)}$ is described in the following section. We present the result

$$\begin{aligned}&P_{33}^{(6,1)}\Big|_{\eta=\pm 1/2} =\\ &= \pm\left\{-\frac{\lambda'+2\mu}{12}\epsilon_l^3\epsilon_2 k_0^4\left[\tilde{u}_3^{(2,1)} - \left(1+\frac{\lambda'}{2\mu}\right)\left(\frac{\epsilon_2 k_0}{2}\right)^2\tilde{u}_3^{(0,1)}\right] + \frac{g_s^{(4)}\epsilon_1\epsilon_2}{2}\left|\tilde{u}_3^{(0,1)}\right|^2\tilde{u}_3^{(0,1)}\right\}\end{aligned} \tag{3.23}$$

The parameter $g_s^{(4)}$ is an integral characteristic of a three-layer medium. It is associated with the surface forces acting on the plate as viewed from the supports, which lead to an indirect interaction of the transverse modes in the plate.

The conditions for boundary-value problems (3.21)–(3.23) to be solvable are:

$$\mu\epsilon_1^4\partial_T^2\tilde{u}_1^{(0,0)} = \partial_X\{\sigma_{11}^{(4,0)} + (\epsilon_1\epsilon_2)^2 q\left|\tilde{u}_3^{(0,1)}\right|^2\} \tag{3.24}$$

$$\begin{aligned}\mu\epsilon_1^4\partial_T^2\tilde{u}_3^{(0,1)} &= -k_0^2\sigma_{11}^{(4,0)}\tilde{u}_3^{(0,1)} + -\left[p - \frac{(\lambda'+2\mu)^2}{48\mu}\right]\epsilon_2^4 k_0^6\tilde{u}_3^{(0,1)} +\\ &+ \frac{\lambda'+2\mu}{4}(\epsilon_1\epsilon_2 k_0)^2\partial_X^2\tilde{u}_3^{(0,1)} + g_{sv}^{(4)}\left|\tilde{u}_3^{(0,1)}\right|^2\tilde{u}_3^{(0,1)}\end{aligned} \tag{3.25}$$

$$q = k_0^4\left[\left(1+\frac{\lambda'}{2\mu}\right)\left(\frac{2}{3}\mu - \frac{\lambda'}{4}\right) + \frac{3a_1+a_2}{6}\right],\quad g_{sv}^{(4)} = g_s^{(4)} + g_v^{(4)}$$

Equations (3.14), (3.24) and (3.25) from a closed system for the calculating the fields $\sigma_{11}^{(4,0)}$, $\bar{u}_3^{(0,1)}$, $\bar{u}_1^{(0,0)}$. Its solutions correspond to different initial conditions and different methods of loading the plate when $|X| \to \infty$.

When the deformation of the plate is homogeneous at infinity, it is necessary to put

$$\begin{aligned}\sigma_{11}^{(4,0)}\Big|_{|X|\to\infty} &= [T_{11}^{\text{ext}}]^{(4)} - \left[\frac{1}{2\mu} + \frac{3a_1+a_2}{(\lambda'+2\mu)^2}\right][T_{11}^{\text{lin}}]^2\\ \tilde{u}_3^{(0,1)}\Big|_{|X|\to\infty} &= \partial_T\tilde{u}_1^{(0,0)}\Big|_{|X|\to\infty} = 0\end{aligned} \tag{3.26}$$

In all, it is simpler to obtain this expression for $\sigma_{11}^{(4,0)}|_{|X|\to\infty}$ by the method used in [2].

Perturbation theory for the supports. Other slow coordinates are used to calculate the displacements $\bar{v}_i$, which better reflect the space-time response of the semi-infinite supports to the bending of the plate.

The calculation of the components $\bar{\upsilon}_i^{(n)}$ in the expansion (1.8) reduces to the recurrent solution of quasistatic boundary-value problems of the linear theory of elasticity, in which the slow time occurs as a parameter. In this case, the equations defining the fields $\bar{\upsilon}_i^{(0)}$ are homogeneous and the equations for calculating the functions $\bar{\upsilon}_i^{(n)}$ with $n \geq 1$ contain bulk forces which are induced by the displacements found in the preceding orders of perturbation theory. Specification of the form of solution (3.1), which involves separating out the resonance mode and introducing the slow coordinate X, enables one to avoid the appearance of complex integrals in the construction of the simplified model. The boundary-value problems for calculating the components $\bar{\upsilon}_i^{(n,\,l)}$ and $l \neq 0$ are Fourier transforms with respect to the variable ξ_1 of linear boundary-value problems with sources.

When $n = 0, l \geq 0$, the displacements $\bar{\upsilon}_i^{(n,\,l)}$ with $l = 1$ are not equal to zero:

$$\mathbf{v}^{(0,\,1)} = \tilde{u}_3^{(0,\,1)}\mathbf{s}$$

$$\mathbf{s} = \frac{\exp(-|k_0|\xi_3^\pm)}{\gamma+1}\left\|\begin{matrix} i\,\mathrm{sign}\,\xi_3^\pm\,\mathrm{sign}\,k_0[\gamma-1-2|k_0\xi_3^\pm|] \\ \gamma+1+2|k_0\xi_3^\pm| \end{matrix}\right\|, \quad \gamma = \frac{\breve{\lambda}+3\breve{\mu}}{\breve{\lambda}+\breve{\mu}} \tag{3.27}$$

The boundary values (3.16), which were used in the preceding section, are calculated using relations (3.27).

We will illustrate the general scheme of integrations, taking as an example the second-order equations of perturbation theory

$$ik_0 l\breve{P}_{i1}^{(2,\,l)} + \epsilon_1\partial_X\breve{P}_{i1}^{(1,\,l)} + \partial_3\breve{P}_{i3}^{(2,\,l)} = 0 \tag{3.28}$$

The boundary conditions for these equations are determined from conditions (2.1) and (2.3)

$$\bar{\upsilon}_3^{(1,\,l)}\big|_{\xi_3^\pm=0} = 0;\quad \breve{P}_{13}^{(2,\,l)}\big|_{\xi_3^\pm=0} = 0,\quad l\neq 2$$
$$\breve{P}_{13}^{(2,\,2)}\big|_{\xi_3^\pm=0} = \pm\frac{\breve{\lambda}'+2\breve{\mu}}{2}\epsilon_1^2 ik_0|k_0|(\tilde{u}_3^{(0,\,1)})^2 \tag{3.29}$$

Problem (3.28), (3.29) has non-zero solutions only for $\mathbf{v}^{(1,\,l)}$ when $|l| \leq 2$.

When $l = 0$, system (3.28) reduces to the equations $\partial_3\breve{P}_{i3}^{(2,\,0)} = 0$, and it follows from these that $\breve{P}_{i3}^{(2,\,0)} = \mathrm{const}$. The constants of integration are assumed to be zero in order to satisfy conditions (3.29) and the conditions that there are no stresses when $\xi_3 \to \pm\infty$. The displacements $\breve{P}_{i3}^{(2,\,0)} = 0$ are calculated from the equations $\mathbf{v}^{(1,\,0)}$

$$\bar{\upsilon}_3^{(1,\,0)} = -\frac{8\epsilon_1 k_0^2}{(\gamma+1)^3}\int_0^{\xi_3^\pm}|\chi^{(0,\,1)}|^2\{\gamma^2+1+2(\gamma+\alpha_1)[(\gamma-1)|k_0\xi_3|+2(k_0\xi_3)^2]+$$
$$+(\gamma-1)^2(\alpha_2+1)\}d\xi_3 \tag{3.30}$$
$$\bar{\upsilon}_1^{(1,\,0)} = 0$$

where

$$\alpha_1 = \frac{\breve{A}+2\breve{B}}{\breve{\lambda}+\breve{\mu}},\quad \alpha_2 = \frac{3\breve{B}+2\breve{C}+\breve{A}/2}{\breve{\lambda}+\breve{\mu}}$$
$$\chi^{(0,\,1)} = \tilde{u}_3^{(0,\,1)}\exp(-|k_0|\xi_3)$$

For the subsequent analysis, it is useful to introduce the vectors $\mathbf{v}^{(n,\,l)} = (\bar{\upsilon}_1^{(n,\,l)}, \bar{\upsilon}_3^{(n,\,l)})$ and the matrix operator

$$\hat{\mathbf{H}}_l = \left\|\begin{matrix} \breve{\mu}\partial_3^2-(\breve{\lambda}+2\breve{\mu})(k_0 l)^2 & (\breve{\lambda}+\breve{\mu})i(k_0 l)\partial_3 \\ (\breve{\lambda}+\breve{\mu})i(k_0 l)\partial_3 & (\breve{\lambda}+2\breve{\mu})\partial_3^2-\breve{\mu}(k_0 l)^2 \end{matrix}\right\| \tag{3.31}$$

Then, when $l = 1$, system (3.28) can be represented in the form

$$\hat{\mathbf{H}}_1\mathbf{v}^{(1,1)} - i\epsilon_1(\partial_{k_0}\hat{\mathbf{H}}_1)\partial_X\mathbf{v}^{(0,1)} = 0$$

It is easy to show that

$$\mathbf{v}^{(1,1)} = -i\epsilon_1\partial_{k_0}\partial_X\mathbf{v}^{(0,1)}; \quad \check{P}_{ij}^{(2,1)} = -i\epsilon_1\partial_{k_0}\partial_X\check{P}_{ij}^{(1,1)} \tag{3.32}$$

The values of $\check{P}_{33}^{(2,1)}|_{\xi_3^{\pm}=0}$ were used in the preceding section when analysing the dynamics of the plate.

When $l = 2$, system (3.28) has the form

$$\hat{\mathbf{H}}_2\mathbf{v}^{(1,2)} = \mathbf{f}^{(1,2)}; \quad \mathbf{f}^{(1,2)} = -\frac{1}{\epsilon_1}\begin{Vmatrix}2ik_0\pi_{11}^{(2,2)} + \partial_3\pi_{13}^{(2,2)}\\ 2ik_0\pi_{31}^{(2,2)} + \partial_3\pi_{33}^{(2,2)}\end{Vmatrix}$$

where $\pi_{ij}^{(2,2)}$ is the non-linear part of the tensor $\check{P}_{ij}^{(2,2)}$ which is expressed in terms of already known fields ($\pi_{ij}^{(2,2)} \sim [\chi^{(0,1)}]^2$). The solution which satisfies conditions (3.29) and the requirement that $\mathbf{v}^{(1,2)} \to \infty$ when $|\xi_3| \to \infty$ has the form

$$\begin{Vmatrix}\bar{v}_1\\ \bar{v}_3\end{Vmatrix}^{(1,2)} = -\frac{\epsilon_1 k_0(\gamma+Q)}{(\gamma+1)^2}[\chi^{(0,1)}]^2\begin{Vmatrix}i[1+2|k_0\xi_3^{\pm}|]\\ -2k_0\xi_3^{\pm}\end{Vmatrix} \tag{3.33}$$

where

$$Q = \gamma^2 + 2\gamma - 2 - (\gamma-1)^2\alpha_2 + (\gamma-1)\alpha_1$$

The function $\check{P}_{33}^{(2,2)}$ and its boundary values

$$\check{P}_{33}^{(2,2)}\big|_{\xi_3^{\pm}=0} = 2\frac{\check{\lambda}+\check{\mu}}{(\gamma+1)^2}(\epsilon_1 k_0)^2(\bar{u}_3^{(0,1)})^2\{2Q + (\gamma-1)^2(\alpha_2+1)\}$$

which are needed when formulating the boundary conditions in the next higher order of perturbation theory, are calculated from relations (3.33) and (1.4).

In order to construct the model of the bending of the plate, it remains to calculate the boundary values of the function $\check{P}_{33}^{(3,1)}$. It is noteworthy that it is not obligatory to calculate the displacements $\mathbf{v}^{(2,1)}$ in order to do this.

In order to find $\check{P}_{33}^{(3,1)}|_{\xi_3^{\pm}=0}$, we return to the third-order equations of perturbation theory with $l = 1$

$$\hat{\mathbf{H}}_1\mathbf{w} + \frac{1}{\epsilon_1}(ik_0\mathbf{a} + \partial_3\mathbf{b}) = 0 \tag{3.34}$$

where

$$\mathbf{w} = \mathbf{v}^{(2,1)} + \frac{1}{2}\epsilon_1^2\partial_{k_0}^2\partial_X^2\mathbf{v}^{(0,1)}, \quad \mathbf{a} = (\pi_{11}^{(3,1)}, \pi_{31}^{(3,1)})^T, \quad \mathbf{b} = (\pi_{13}^{(3,1)}, \pi_{33}^{(3,1)})^T$$

and $\pi_{ij}^{(3,1)}$ is the non-linear part of the tensor $\check{P}_{ij}^{(3,1)}$, which is expressed in terms of the fields ($\pi_{ij}^{(3,1)} \sim [\chi^{(0,1)}]^2\chi^{(0,1)}$) that have already been found.

We multiply Eq. (3.34) scalarly from the left by the vector-function $\mathbf{p} = \mathbf{s}^T\boldsymbol{\sigma}_3 = (s_1, -s_3)$ (see relation (3.27)), which is the solution of the adjoint boundary-value problem (it satisfies the equation $\mathbf{p}\hat{\mathbf{H}}_1^+ = 0$, where $\hat{\mathbf{H}}_1^+$ is an operator which is the Hermitian conjugate of $\hat{\mathbf{H}}_1$). The result is integrated over the domain $\Gamma^+ = \{\xi_3^+ \geq 0\}$ ($\Gamma^- = \{\xi_3^- \leq 0\}$). After some simple reduction, taking account of the equality

$$\partial_{k_0}\mathbf{v}^{(0,1)}\big|_{\xi_3^{\pm}=0} = \partial_{k_0}^2\mathbf{v}^{(0,1)}\big|_{\xi_3^{\pm}=0} = 0$$

we obtain an integral representation for the boundary values of the function $\check{P}_{33}^{(3,1)}$

$$\breve{P}_{33}^{(3,1)}\Big|_{\xi_3^\pm=0} = \pm\left\{\int_{\Gamma^\pm}[\partial_3\mathbf{p}\cdot\mathbf{b}-ik_0(\mathbf{p}\cdot\mathbf{a})]d\xi_3 + i\,\mathrm{sign}\,k_0\left(\frac{\gamma-1}{\gamma+1}\right)\breve{P}_{13}^{(3,1)}\Big|_{\xi_3^\pm=0} - \right.$$
$$\left. -\frac{\breve{\lambda}'+2\breve{\mu}}{2}\epsilon_1|k_0|\bar{\upsilon}_3^{(2,1)}\Big|_{\xi_3^\pm=0}\right\} \tag{3.35}$$

Fields which have already been calculated occur on the right-hand side of equality (3.35). In particular, the components $\breve{P}_{13}^{(3,1)}|_{\xi_3^\pm=0}$ are determined from conditions (2.3), and $\bar{\upsilon}_3^{(2,1)}|_{\xi_3^\pm=0}$ are determined from conditions (2.1) and relations (3.9).

In the final analysis, formula (3.23), which was used in the preceding section when analysing the dynamics of the plate, follows from conditions (2.4) and (3.35). The parameter $g_s^{(4)}$ in relation (3.23) is determined from the equality

$$\frac{\epsilon_1\epsilon_2 g_s^{(4)}}{2} = 4(\breve{\lambda}+\breve{\mu})\frac{(\gamma-1)}{(\gamma+1)^3}\left[Q-\frac{5}{4}(\gamma-1)^2+2\gamma\right](\epsilon_1|k_0|)^3 +$$
$$+\int_{\Gamma^\pm}\frac{(\partial_3\mathbf{p}\cdot\mathbf{b})-ik_0(\mathbf{p}\cdot\mathbf{a})}{|\tilde{u}_3^{(0,1)}|^2\tilde{u}_3^{(0,1)}}d\xi_3 \tag{3.36}$$

The explicit dependence of $g_s^{(4)}$ on the elastic moduli of the three-layer medium has the form

$$\frac{\epsilon_1\epsilon_2 g_s^{(4)}}{2} = 2\frac{\breve{\lambda}+\breve{\mu}}{(\gamma+1)^5}(\epsilon_1|k_0|)^3\left[-\left\{\frac{9}{2}+4\alpha_2+\beta_3\right\}(\gamma-1)^5 +\right.$$
$$+2\{(3+\alpha_2(3+3\alpha_2))-\beta_3\}(\gamma-1)^4+2\{18+Q+4\alpha_1+\alpha_2(2-Q+2\alpha_1)-\beta_2\}(\gamma-1)^3+$$
$$+2\{35+Q(7-2\alpha_2+\alpha_1)+2\alpha_2(1+\alpha_1)+\alpha_1(11+\alpha_1)-2\beta_2\}(\gamma-1)^2+ \tag{3.37}$$
$$\left.+\{31+4Q(6+\alpha_1)+2\alpha_1(13+3\alpha_1)-6\beta_1\}(\gamma-1)+2\{4Q+3\alpha_1^2-6\beta_1\}\right]$$

where

$$\beta_1 = \frac{\breve{H}}{\breve{\lambda}+\breve{\mu}},\quad \beta_2 = \frac{\breve{F}+\breve{H}+2\breve{M}}{\breve{\lambda}+\breve{\mu}},\quad \beta_3 = \frac{\breve{F}+\frac{3}{2}\breve{H}+6\breve{M}+2\breve{N}}{\breve{\lambda}+\breve{\mu}}$$

4. SOLITONS OF THE CORRUGATION OF THE MIDDLE LAYER

We shall seek a solution of system (3.14), (3.24), (3.25) in the form

$$\tilde{u}_1^{(0,0)} = \frac{T_{11}^{\mathrm{lin}}}{\varepsilon_1^2(\lambda'+2\mu)}X - k_0^2\int^{X+VT}A^2(X')dX'$$
$$\tilde{u}_3^{(0,1)} = A(X+VT)\exp(i\Omega T+i\kappa X+i\phi_0) \tag{4.1}$$

where V, κ, Ω and ϕ_0 are real parameters. From Eqs (3.14) and (3.24), we obtain

$$\sigma_{11}^{(4,0)} = c^{(4)} - |\tilde{u}_3^{(0,1)}|^2[(\epsilon_1\epsilon_2)^2q+\mu\epsilon_1^4(k_0V)^2] \tag{4.2}$$

The constant of integration $c^{(4)}$ is determined by the boundary conditions. In particular, when the deformations of the plate are homogeneous at infinity, it follows from relations (3.26) that

$$c^{(4)} = [T_{11}^{\text{ext}}]^{(4)} - \left\{\frac{1}{2\mu} + \frac{3a_1 + a_2}{\lambda' + 2\mu}\right\}[T_{11}^{\text{lin}}]^2$$

After substituting expressions (4.2) into (3.25), we find the relation between the parameters V, κ and Ω

$$\kappa = \frac{V\Omega}{V_{\text{cr}}^2}, \quad V_{\text{cr}}^2 = \frac{1}{2}\left(\frac{\epsilon_2 k_0}{\epsilon_1}\right)^2\left(1 + \frac{\lambda'}{2\mu}\right) \tag{4.3}$$

and the ordinary differential equation for determining A, which admits of a first integral

$$(\partial_X A)^2 = \alpha A^2 + \frac{\beta}{2}A^4 + c \tag{4.4}$$

Here,

$$\alpha = -k_0^2[\mu\epsilon_1^4(V^2 - V_{\text{cr}})^2]^{-1}\left\{(\epsilon_2 k_0)^4\left(p - \frac{\lambda' + 2\mu}{48\mu}\right) + \mu\epsilon_1^4\left(\frac{\Omega}{V_{\text{cr}}k_0}\right)^2(V^2 - V_{\text{cr}}^2) + c^{(4)}\right\}$$

$$\beta = [\mu\epsilon_1^4(V^2 - V_{\text{cr}}^2)]^{-1}\{g_{\text{sv}}^{(4)} + (k_0\epsilon_1\epsilon_2)^2 q + \mu(\epsilon_1^2 V k_0^2)^2\}$$

and c is a constant of integration.

In the case when the boundary conditions (3.26) are homogeneous at infinity, a localized solution of Eq. (4.4) exists when $c = 0$, $\alpha > 0$, $\beta < 0$, and it is soliton of the transverse corrugation of the plate

$$A = \frac{\sqrt{2\alpha/|\beta|}}{\cosh(\sqrt{\alpha}[X + VT])} \tag{4.5}$$

According to expression (4.5), there are non-zero deflections of the plate in a domain with a characteristic dimension of the order of $\alpha^{-1/2}$, which moves with a velocity V. When $\Omega = \kappa = 0$, the transverse displacements of the plate have the form

$$\bar{u}_3^{(1)} = A\cos(k_0\xi_1 + \phi_0)$$

To be specific, suppose $g_{\text{sv}}^{(4)} > 0$. Then, the soliton (4.1), (4.5) is formed in the case of loads T_{11}^{ext} which are smaller than a certain critical value (close to T_{11}^{lin}):

$$|T^{\text{ext}}| < |T^{\text{lin}}|\left\{1 - \frac{2}{3}\left(\frac{\epsilon_2 k_0}{2}\right)^2\left[\frac{17}{5} + \frac{\lambda'}{\mu} + \frac{3a_1 + a_2}{\lambda' + 2\mu}\right]\right\} + \mu\left[\frac{\Omega\epsilon_1^2}{V_{\text{cr}}k_0}\right]^2(V^2 - V_{\text{cr}}^2) \tag{4.6}$$

and moves with a velocity which does not exceed $V_{\text{cr}}(V^2 < V_{\text{cr}}^2)$. In particular, such solitons can also be stationary. They are stress concentrators and, simultaneously, the forerunners of the subsequent plastic deformation of the material.

When $c > 0$, $0 > c > -\alpha/(2|\beta|)$, $\alpha > 0$, $\beta < 0$, bounded solutions exist which describe, in particular, structures in the form of chains of corrugation solitons

$$A = A_+\text{dn}\left\{A_+\sqrt{\frac{|\beta|}{2}}(X + VT), k\right\}, \quad k^2 = \frac{A_+^2 - A_-^2}{A_+^2}, \quad 0 > c > -\frac{\alpha^2}{2|\beta|} \tag{4.7}$$

$$A = A_+\text{cn}\left\{\frac{A_+}{k}\sqrt{\frac{|\beta|}{2}}(X + VT), k\right\}, \quad k^2 = \frac{A_+^2}{A_+^2 + |A_-|^2}, \quad c > 0 \tag{4.8}$$

Here

$$A_{\pm} = \sqrt{(\alpha \pm D)/|\beta|}, \quad D = \sqrt{\alpha^2 + 2c|\beta|}$$

We shall discuss the form of the solution (4.7) for values of the parameter k close to unity. We will use the representation

$$\mathrm{dn}(Y,k) = \frac{\pi}{2K'}\sum_{n=-\infty}^{+\infty} \mathrm{sech}\left[\frac{\pi}{2K'}(Y - 2Kn)\right] \tag{4.9}$$

where $K = K(k)$, $K' = K(k')$ are complete elliptic integrals of the first kind, $k' = \sqrt{1-k^2}$ ($k' \ll 1$) and n integers. Relation (4.9) is proved by comparing the expansions in common fractions [6] of the left-hand and right-hand sides of formula (4.9). The additive constant is fixed by the normalization condition: $\mathrm{dn}(Y = 0, k) = 1$ and turns out to be zero.

According to relations (4.7) and (4.9) when $k' \ll 1$, changes in the function A are localized close to points with coordinates $X = (2Kn/A_+)\sqrt{2/|\beta|}$. In the neighbourhood of each of these points, in the domain with a characteristic dimension $l_0 = (2K'/(\pi A_+))\sqrt{2/|\beta|}$, the solution of (4.1), (4.7) appears as a corrugation soliton. Outside these domains the deformations of the material are small.

When $\alpha < 0$, $\beta > 0$, bounded, "quasiperiodic" (4.1) solutions exist when $0 < c \le \alpha^2/(2\beta)$:

$$A = A_-\mathrm{sn}\left\{A_+\sqrt{\frac{\beta}{2}}(X + VT), k\right\}, \quad k = \frac{A_-}{A_+} \tag{4.10}$$

Here,

$$A_{\pm} = \sqrt{(|\alpha| \pm D)/\beta}, \quad D = \sqrt{\alpha^2 - 2c\beta}$$

The simplest of the solutions is obtained when $c = \alpha^2/(2\beta)$ and this is a so-called "dark" soliton

$$A = \sqrt{\frac{|\alpha|}{\beta}}\,\mathrm{th}\left(\sqrt{\frac{|\alpha|}{2}}[X + VT]\right) \tag{4.11}$$

An excited state of the plate with an asymptotic form of the transverse wave type

$$\tilde{u}_3^{(1)} \sim \cos(k_0\xi_1 + \kappa X + \Omega T + \phi_0), \quad X \to \pm\infty$$

corresponds to the soliton of (4.1), (4.11).

The "dark" soliton (4.11) describes the modulation of this wave. When $g_{\mathrm{sv}}^{(4)} > 0$, "dark" solitons are formed below the threshold of stability of the plate and move with velocities V which exceed $|V_{\mathrm{cr}}|$ in absolute value.

5. CONCLUSION

The difficulty in the theoretical description of the non-linearly elastic dynamics of a three-layer medium are not solely due to the non-linearity of the material. The non-local nature of the interaction between the layers makes the problem significantly more difficult. In addition, it is important, in the case of a layered medium, to take account of the boundary conditions accurately, since they determine the dispersion of the medium, and this also means the conditions for the formation of soliton-like excitations. Long-lived, non-linearly elastic solitons are promising for the diagnostics of structurally inhomogeneous materials since they carry useful information both on the stressed state of a medium as well as on the geometrical dimensions and material parameters of its individual layers.

The simplest case has been considered above when the external stress and the boundary conditions in the x_1Ox_2 plane lead to the quasi-one-dimensional corrugation of one of the layers of the material, which is the result of the interaction of the neutral-stable linear mode $\sim\exp(ik_0\xi_1)$ with close unstable modes. In the general case, it is necessary to take account of the interaction of several groups of waves in order to describe the bending of the layers in several directions. It is necessary to modify perturbation theory in the case of other boundary conditions and material parameters of a layered medium. This leads to a change in the effective model, but the scheme for constructing it remains essentially unchanged.

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